

GROUP ACTIONS ON PRO- C^* -ALGEBRAS AND THEIR PRO- C^* -CROSSED PRODUCTS

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ABSTRACT. In this paper, we define the notions of full pro- C^* -crossed product, respectively reduced pro- C^* -crossed product, of a pro- C^* -algebra $A[\tau_\Gamma]$ by a strong bounded action α of a locally compact group G and investigate some of their properties.

1. INTRODUCTION

Given a C^* -algebra A and a continuous action α of a locally compact group G on A , we can construct a new C^* -algebra, called the crossed product of A by α , usually denoted by $G \times_\alpha A$, and which contains, in some subtle sense, A and G . The origin of this construction goes back to Murray and von Neumann and their group measure space construction by which they associated a von Neumann algebra to a countable group acting on a measure space. The analog of this construction for the case of C^* -algebras is due to Gelfand with co-authors Naimark and Fomin. There is a vast literature on crossed products of C^* -algebras (see, for example, [W]), but the corresponding theory in the context of non-normed topological $*$ -algebras has still a long way to go.

Crossed product of pro- C^* -algebras by inverse limit actions of locally compact groups were considered by Phillips [P2] and JoiȚa [J2, J3, J4]. If α is an inverse limit action of a locally compact group G on a pro- C^* -algebra $A[\tau_\Gamma]$ whose topology is given by the family of C^* -seminorms $\Gamma = \{p_\lambda\}_{\lambda \in \Lambda}$, then the covariance algebra $L^1(G, \alpha, A[\tau_\Gamma])$ has a structure of locally m -convex $*$ -algebra with topology given by the family of submultiplicative seminorms $\{N_{p_\lambda}\}_{\lambda \in \Lambda}$, where

$$N_{p_\lambda}(f) = \int_G p_\lambda(f(g)) dg$$

and the full crossed product of $A[\tau_\Gamma]$ by α is defined as the enveloping pro- C^* -algebra of $L^1(G, \alpha, A[\tau_\Gamma])$. In particular, for a given inverse limit automorphism α of a pro- C^* -algebra $A[\tau_\Gamma]$, we can associate to the pair $(A[\tau_\Gamma], \alpha)$ a pro- C^* -algebra by the crossed product construction, but if α is not an inverse limit automorphism, this construction is not possible because the covariance algebra has not a structure of locally m -convex $*$ -algebra (N_{p_λ} is not a submultiplicative $*$ -seminorm). On the other hand, a transformation group (X, G) , with X a countably compactly generated Hausdorff topological space (X is a direct limit of a countable family of compact spaces $\{K_n\}_n$), induces an action α of G on the pro- C^* -algebra $C(X)$,

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which is not in general an inverse limit action and so, we can not associate to (X, G) a pro- C^* -algebra by the above construction.

It is well known that the crossed product of C^* -algebras is a universal object for nondegenerate covariant representations (see, for example, [R]). In [J3], we show that the crossed product of pro- C^* -algebras by inverse limit actions has also the universal property with respect to the nondegenerate covariant representations. In this paper, we define the full crossed product of a pro- C^* -algebra $A[\tau_\Gamma]$ by an action α of a locally compact group G as a universal object for nondegenerate covariant representations and we show that the full crossed product of pro- C^* -algebras exists for strong bounded actions. Strong boundless of the action α is essential to prove the existence of a covariant representation. Unfortunately, if the action α of G on $A[\tau_\Gamma]$ is strongly bounded, then there is another family of C^* -seminorms on $A[\tau_\Gamma]$ which induces the same topology on A , and α is an inverse limit action with respect to this family of C^* -seminorms.

The organization of this paper is as follows. After preliminaries in Section 2, we introduce the notion of strong bounded action of a locally compact group G on a pro- C^* -algebra $A[\tau_\Gamma]$ and present some examples in Section 3. We show that there is a nondegenerate covariant representation of a pro- C^* -algebra $A[\tau_\Gamma]$ with respect to a strong bounded action α of a locally compact group G on $A[\tau_\Gamma]$ in Section 4. In Section 5, we show that if α is strongly bounded, then there exists the full pro- C^* -crossed product of $A[\tau_\Gamma]$ by α . Also, we show that the full pro- C^* -crossed product is invariant under the conjugacy of the actions, and the full pro- C^* -crossed product of the maximal tensor product $A[\tau_\Gamma] \otimes_{\max} B[\tau_{\Gamma'}]$ by the action $\alpha \otimes \text{id}$ is isomorphic to the maximal tensor product of the full pro- C^* -crossed product of $A[\tau_\Gamma]$ by α and $B[\tau_{\Gamma'}]$. In Section 6, we define the notion of reduced pro- C^* -crossed product of a pro- C^* -algebra $A[\tau_\Gamma]$ by a strong bounded action α , and show that the reduced pro- C^* -crossed product is invariant under the conjugacy of the actions. Also, we show that the reduced pro- C^* -crossed product of the minimal tensor product $A[\tau_\Gamma] \otimes_{\min} B[\tau_{\Gamma'}]$ by the action $\alpha \otimes \text{id}$ is isomorphic to the minimal tensor product of the reduced pro- C^* -crossed product of $A[\tau_\Gamma]$ by α and $B[\tau_{\Gamma'}]$. It is known that the full crossed product of a C^* -algebra by an action α of an amenable locally compact group G coincides with the reduced crossed product. We show that this result is still valid for pro- C^* -crossed products.

2. PRELIMINARIES

A seminorm p on a topological $*$ -algebra A satisfies the C^* -condition (or is a C^* -seminorm) if $p(a^*a) = p(a)^2$ for all $a \in A$. It is known that such a seminorm must be submultiplicative ($p(ab) \leq p(a)p(b)$ for all $a, b \in A$) and $*$ -preserving ($p(a^*) = p(a)$ for all $a \in A$).

A pro- C^* -algebra is a complete Hausdorff topological $*$ -algebra A whose topology is given by a directed family of C^* -seminorms $\{p_\lambda; \lambda \in \Lambda\}$. Other terms used for pro- C^* -algebras are: locally C^* -algebras (A. Inoue, M. Fragouloupoulou, A. Mallios, etc.), LMC^* -algebras (G. Lassner, K. Schmüdgen), b^* -algebras (C. Apostol).

Let $A[\tau_\Gamma]$ be a pro- C^* -algebra with topology given by $\Gamma = \{p_\lambda; \lambda \in \Lambda\}$ and let $B[\tau_{\Gamma'}]$ be a pro- C^* -algebra with topology given by $\Gamma' = \{q_\delta; \delta \in \Delta\}$. A continuous $*$ -morphism $\varphi : A[\tau_\Gamma] \rightarrow B[\tau_{\Gamma'}]$ (that is, φ is linear, $\varphi(ab) = \varphi(a)\varphi(b)$ and $\varphi(a^*) = \varphi(a)^*$ for all $a, b \in A$ and for each $q_\delta \in \Gamma'$, there is $p_\lambda \in \Gamma$ such that $q_\delta(\varphi(a)) \leq p_\lambda(a)$ for all $a \in A$) is called a pro- C^* -morphism. The pro- C^* -algebras $A[\tau_\Gamma]$ and

$B[\tau_{\Gamma'}]$ are isomorphic if there is a pro- C^* -isomorphism $\varphi : A[\tau_{\Gamma}] \rightarrow B[\tau_{\Gamma'}]$ (that is, φ is invertible, φ and φ^{-1} are pro- C^* -morphisms).

If $\{A_{\lambda}; \pi_{\lambda\mu}\}_{\lambda \geq \mu, \lambda, \mu \in \Lambda}$ is an inverse system of C^* -algebras, then $\lim_{\leftarrow \lambda} A_{\lambda}$ with topology given by the family of C^* -seminorms $\{p_{\lambda}\}_{\lambda \in \Lambda}$, with $p_{\lambda}\left((a_{\mu})_{\mu \in \Lambda}\right) = \|a_{\lambda}\|_{A_{\lambda}}$ for all $\lambda \in \Lambda$, is a pro- C^* -algebra.

Let $A[\tau_{\Gamma}]$ be a pro- C^* -algebra with topology given by $\Gamma = \{p_{\lambda}; \lambda \in \Lambda\}$. For $\lambda \in \Lambda$, $\ker p_{\lambda}$ is a closed $*$ -bilateral ideal and $A_{\lambda} = A/\ker p_{\lambda}$ is a C^* -algebra in the C^* -norm $\|\cdot\|_{p_{\lambda}}$ induced by p_{λ} (that is, $\|a\|_{p_{\lambda}} = p_{\lambda}(a)$, for all $a \in A$). The canonical map from A to A_{λ} is denoted by π_{λ}^A , $\pi_{\lambda}^A(a) = a + \ker p_{\lambda}$ for all $a \in A$. For $\lambda, \mu \in \Lambda$ with $\mu \leq \lambda$ there is a surjective C^* -morphism $\pi_{\lambda\mu}^A : A_{\lambda} \rightarrow A_{\mu}$ such that $\pi_{\lambda\mu}^A(a + \ker p_{\lambda}) = a + \ker p_{\mu}$, and then $\{A_{\lambda}; \pi_{\lambda\mu}^A\}_{\lambda, \mu \in \Lambda}$ is an inverse system of C^* -algebras. Moreover, pro- C^* -algebras $A[\tau_{\Gamma}]$ and $\lim_{\leftarrow \lambda} A_{\lambda}$ are isomorphic (Arens-Michael decomposition).

Let $\{(\mathcal{H}_{\lambda}, \langle \cdot, \cdot \rangle_{\lambda})\}_{\lambda \in \Lambda}$ be a family of Hilbert spaces such that $\mathcal{H}_{\mu} \subseteq \mathcal{H}_{\lambda}$ and $\langle \cdot, \cdot \rangle_{\lambda}|_{\mathcal{H}_{\mu}} = \langle \cdot, \cdot \rangle_{\mu}$ for all $\lambda, \mu \in \Lambda$ with $\mu \leq \lambda$. $\mathcal{H} = \lim_{\lambda \rightarrow} \mathcal{H}_{\lambda}$ with inductive limit topology is called a locally Hilbert space.

Let $L(\mathcal{H}) = \{T : \mathcal{H} \rightarrow \mathcal{H}; T_{\lambda} = T|_{\mathcal{H}_{\lambda}} \in L(\mathcal{H}_{\lambda}) \text{ and } P_{\lambda\mu}T_{\lambda} = T_{\lambda}P_{\lambda\mu} \text{ for all } \lambda, \mu \in \Lambda \text{ with } \mu \leq \lambda\}$, where $P_{\lambda\mu}$ is the projection of \mathcal{H}_{λ} on \mathcal{H}_{μ} . Clearly, $L(\mathcal{H})$ is an algebra in an obvious way, and $T \rightarrow T^*$ with $T^*|_{\mathcal{H}_{\lambda}} = (T_{\lambda})^*$ for all $\lambda \in \Lambda$ is an involution.

For each $\lambda \in \Lambda$, the map $p_{\lambda, L(\mathcal{H})} : L(\mathcal{H}) \rightarrow [0, \infty)$ given by $p_{\lambda, L(\mathcal{H})}(T) = \|T|_{\mathcal{H}_{\lambda}}\|_{L(\mathcal{H}_{\lambda})}$ is a C^* -seminorm on $L(\mathcal{H})$, and with topology given by the family of C^* -seminorms $\{p_{\lambda, L(\mathcal{H})}\}_{\lambda \in \Lambda}$, $L(\mathcal{H})$ becomes a pro- C^* -algebra.

$L(\mathcal{H})$ as a pro- C^* -algebra has an Arens-Michael decomposition, given by the C^* -algebras $L(\mathcal{H})_{\lambda} = L(\mathcal{H})/\ker p_{\lambda}$, $\lambda \in \Lambda$. Moreover, for each $\lambda \in \Lambda$, the map $\varphi_{\lambda} : L(\mathcal{H})_{\lambda} \rightarrow L(\mathcal{H}_{\lambda})$ given by $\varphi_{\lambda}(T + \ker p_{\lambda}) = T|_{\mathcal{H}_{\lambda}}$ is an isometric $*$ -morphism. The canonical maps from $L(\mathcal{H})$ to $L(\mathcal{H})_{\lambda}$, $\lambda \in \Lambda$ are denoted by $\pi_{\lambda}^{\mathcal{H}}$, $\lambda \in \Lambda$, and $\pi_{\lambda}^{\mathcal{H}}(T) = T|_{\mathcal{H}_{\lambda}}$. For a given pro- C^* -algebra $A[\tau_{\Gamma}]$ there is a locally Hilbert space \mathcal{H} such that $A[\tau_{\Gamma}]$ is isomorphic to a pro- C^* -subalgebra of $L(\mathcal{H})$ (see [I, Theorem 5.1]).

A multiplier of $A[\tau_{\Gamma}]$ is a pair (l, r) of linear maps $l, r : A[\tau_{\Gamma}] \rightarrow A[\tau_{\Gamma}]$ such that are respectively left and right A -module homomorphisms and $r(a)b = al(b)$ for all $a, b \in A$. The set $M(A[\tau_{\Gamma}])$ of all multipliers of $A[\tau_{\Gamma}]$ is a pro- C^* -algebra with multiplication given by $(l_1, r_1)(l_2, r_2) = (l_1l_2, r_2r_1)$, the involution given by $(l, r)^* = (r^*, l^*)$, where $r^*(a) = r(a^*)^*$ and $l^*(a) = l(a^*)^*$ for all $a \in A$, and the topology given by the family of C^* -seminorms $\{p_{\lambda, M(A[\tau_{\Gamma}])}\}_{\lambda \in \Lambda}$, where $p_{\lambda, M(A[\tau_{\Gamma}])}(l, r) = \sup\{p_{\lambda}(l(a)); p_{\lambda}(a) \leq 1\}$. Moreover, for each $p_{\lambda} \in \Gamma$, the C^* -algebras $(M(A[\tau_{\Gamma}]))_{\lambda}$ and $M(A_{\lambda})$ are isomorphic. The strict topology on $M(A[\tau_{\Gamma}])$ is given by the family of seminorms $\{p_{\lambda, a}\}_{(\lambda, a) \in \Lambda \times A}$, where $p_{\lambda, a}(l, r) = p_{\lambda}(l(a)) + p_{\lambda}(r(a))$, $M(A[\tau_{\Gamma}])$ is complete with respect to the strict topology and $A[\tau_{\Gamma}]$ is dense in $M(A[\tau_{\Gamma}])$ (see [P1] and [J1, Proposition 3.4]).

A pro- C^* -morphism $\varphi : A[\tau_{\Gamma}] \rightarrow M(B[\tau_{\Gamma'}])$ is nondegenerate if $[\varphi(A)B] = B[\tau_{\Gamma'}]$, where $[\varphi(A)B]$ denotes the closed subspace of $B[\tau_{\Gamma'}]$ generated by $\{\varphi(a)b; a \in A, b \in B\}$. A nondegenerate pro- C^* -morphism $\varphi : A[\tau_{\Gamma}] \rightarrow M(B[\tau_{\Gamma'}])$ extends to a unique pro- C^* -morphism $\bar{\varphi} : M(A[\tau_{\Gamma}]) \rightarrow M(B[\tau_{\Gamma'}])$.

3. GROUP ACTIONS ON PRO- C^* -ALGEBRAS

Throughout this paper, $A[\tau_\Gamma]$ is a pro- C^* -algebra with topology given by the family of C^* -seminorms $\Gamma = \{p_\lambda\}_{\lambda \in \Lambda}$ and G is a locally compact group.

Definition 3.1. (1) An action of G on $A[\tau_\Gamma]$ is a group morphism α from G to $\text{Aut}(A[\tau_\Gamma])$ such that the map $t \mapsto \alpha_t(a)$ from G to $A[\tau_\Gamma]$ is continuous for each $a \in A$.

(2) An action α of G on $A[\tau_\Gamma]$ is strongly bounded, if for each $\lambda \in \Lambda$ there is $\mu \in \Lambda$ such that

$$p_\lambda(\alpha_t(a)) \leq p_\mu(a)$$

for all $t \in G$ and for all $a \in A$.

(3) An action α is an inverse limit action, if $p_\lambda(\alpha_t(a)) = p_\lambda(a)$ for all $a \in A$, for all $t \in G$ and for all $\lambda \in \Lambda$.

Remark 3.2. (1) If α is an inverse limit action of G on $A[\tau_\Gamma]$, then for each $\lambda \in \Lambda$, there is an action α^λ of G on A_λ such that $\alpha_t^\lambda \circ \pi_\lambda^A = \pi_\lambda^A \circ \alpha_t$ for all $t \in G$, and then $\alpha_t = \varprojlim_{\lambda} \alpha_t^\lambda$ for all $t \in G$.

(2) Any inverse limit action of G on $A[\tau_\Gamma]$ is strongly bounded.

(3) If A is a C^* -algebra, then any action of G on A is strongly bounded.

(4) If G is a compact group, then any action of G on $A[\tau_\Gamma]$ is strongly bounded.

Let X be a compactly countably generated Hausdorff topological space (that is, X is a direct limit of a countable family $\{K_n\}_n$ of compact spaces). The $*$ -algebra $C(X)$ of all continuous complex valued functions on X is a pro- C^* -algebra with topology given by the family of C^* -seminorms $\{p_{K_n}\}_n$, where $p_{K_n}(f) = \sup\{|f(x)|; x \in K_n\}$.

Example 3.3. Let (G, X) be a transformation group (that is, there is a continuous map $(t, x) \mapsto t \cdot x$ from $G \times X$ to X such that $e \cdot x = x$ and $s \cdot (t \cdot x) = (st) \cdot x$ for all $s, t \in G$ and for all $x \in X$) with $X = \varinjlim_{n \rightarrow} K_n$ a compactly countably generated Hausdorff topological space. Then there is an action α of G on the pro- C^* -algebra $C(X)$, given by

$$\alpha_t(f)(x) = f(t^{-1} \cdot x).$$

If for any positive integer n , there is a positive integer m such that $G \cdot K_n \subseteq K_m$, the action α is strongly bounded, since for each n , there is m such that

$$p_{K_n}(\alpha_t(f)) = \sup\{|f(t^{-1} \cdot x)|; x \in K_n\} \leq \sup\{|f(y)|; y \in K_m\} = p_{K_m}(f)$$

for all $f \in C(X)$ and for all $t \in G$. If $G \cdot K_n = K_n$ for all n , then α is an inverse limit action. Take, for instance, $\mathbb{R} = \varinjlim_{n \rightarrow} [-n, n]$. Suppose that \mathbb{Z}_2 actions on \mathbb{R} by $\widehat{0} \cdot x = x$ and $\widehat{1} \cdot x = 2 - x$ for all $x \in \mathbb{R}$. Then $(\mathbb{Z}_2, \mathbb{R})$ is a transformation group such that for each positive integer n , $\mathbb{Z}_2 \cdot [-n, n] \subseteq [-n - 2, n + 2]$.

Example 3.4. Let $X = \varinjlim_{n \rightarrow} K_n$ be a compactly countably generated Hausdorff topological space and $h : X \rightarrow X$ a homeomorphism with the property that for each positive integer n , there is a positive integer m such that $h^k(K_n) \subseteq K_m$ for all integers k . Then the map $n \mapsto \alpha_n$ from \mathbb{Z} to $\text{Aut}(C(X))$, where $\alpha_n(f) = f \circ h^n$, is a strong bounded action of \mathbb{Z} to $C(X)$. If $h(K_n) = K_n$ for all n , then α is an inverse limit action. Take, for instance, $\mathbb{R} = \varinjlim_{n \rightarrow} [-n, n]$, the map $h : \mathbb{R} \rightarrow \mathbb{R}$ defined by $h(x) = 1 - x$ is a homeomorphism such that for each positive integer n , $h^k([-n, n]) \subseteq [-n - 1, n + 1]$ for all integers k .

Example 3.5. The $*$ -algebra $C[0, 1]$ equipped with the topology 'cc' of uniform convergence on countable compact subsets is a pro- C^* -algebra denoted by $C_{cc}[0, 1]$ (see, for example, [F, p. 104]). The action of \mathbb{Z}_2 on $C_{cc}[0, 1]$ given by $\alpha_{\hat{0}} = id_{C_{cc}[0, 1]}$ and $\alpha_{\hat{1}}(f)(x) = f(1 - x)$ for all $f \in C_{cc}[0, 1]$ and for all $x \in [0, 1]$ is strongly bounded.

Remark 3.6. (1) Let α be a strong bounded action of G on $A[\tau_\Gamma]$. Then, for each $\lambda \in \Lambda$, the map $p^\lambda : A \rightarrow [0, \infty)$ given by

$$p^\lambda(a) = \sup\{p_\lambda(\alpha_t(a)); t \in G\}$$

is a continuous C^* -seminorm on $A[\tau_\Gamma]$. Let $\Gamma^G = \{p^\lambda\}_{\lambda \in \Lambda}$. Since, for each $\lambda \in \Lambda$, there is $\mu \in \Lambda$ such that

$$p_\lambda \leq p^\lambda \leq p_\mu,$$

Γ^G defines on A a structure of pro- C^* -algebra, and moreover, the pro- C^* -algebras $A[\tau_\Gamma]$ and $A[\tau_\Gamma^G]$ are isomorphic.

(2) If the action α of G on $A[\tau_\Gamma]$ is strongly bounded, then α is an inverse limit action of G on $A[\tau_\Gamma^G]$.

4. COVARIANT REPRESENTATIONS

Definition 4.1. A pro- C^* -dynamical system is a triple $(G, \alpha, A[\tau_\Gamma])$, where G is a locally compact group, $A[\tau_\Gamma]$ is a pro- C^* -algebra and α is an action of G on $A[\tau_\Gamma]$.

A representation of a pro- C^* -algebra $A[\tau_\Gamma]$ on a Hilbert space \mathcal{H} is a continuous $*$ -morphism $\varphi : A[\tau_\Gamma] \rightarrow L(\mathcal{H})$. A representation (φ, \mathcal{H}) of $A[\tau_\Gamma]$ is nondegenerate if $[\varphi(A)\mathcal{H}] = \mathcal{H}$.

Definition 4.2. A covariant representation of $(G, \alpha, A[\tau_\Gamma])$ on a Hilbert space \mathcal{H} is a triple $(\varphi, u, \mathcal{H})$ consisting of a representation (φ, \mathcal{H}) of $A[\tau_\Gamma]$ on \mathcal{H} and a unitary $*$ -representation (u, \mathcal{H}) of G on \mathcal{H} such that

$$\varphi(\alpha_t(a)) = u_t \varphi(a) u_t^*$$

for all $a \in A$ and for all $t \in G$. A covariant representation $(\varphi, u, \mathcal{H})$ is nondegenerate if (φ, \mathcal{H}) is nondegenerate.

Two representations $(\varphi, u, \mathcal{H})$ and (ψ, v, \mathcal{K}) of $(G, \alpha, A[\tau_\Gamma])$ are unitarily equivalent if there is a unitary operator $U : \mathcal{H} \rightarrow \mathcal{K}$ such that $U\varphi(a) = \psi(a)U$ for all $a \in A$ and $Uu_t = v_tU$ for all $t \in G$.

For each $p_\lambda \in \Gamma$, we denote by $\mathcal{R}_\lambda(G, \alpha, A[\tau_\Gamma])$ the collection of all equivalence classes of nondegenerate covariant representations $(\varphi, u, \mathcal{H})$ of $(G, \alpha, A[\tau_\Gamma])$ with the property that $\|\varphi(a)\| \leq p_\lambda(a)$ for all $a \in A$. Clearly,

$$\bigcup_{\lambda} \mathcal{R}_\lambda(G, \alpha, A[\tau_\Gamma]) = \mathcal{R}(G, \alpha, A[\tau_\Gamma]),$$

where $\mathcal{R}(G, \alpha, A[\tau_\Gamma])$ denotes the collection of all equivalence classes of nondegenerate covariant representations of $(G, \alpha, A[\tau_\Gamma])$.

Remark 4.3. If α is an inverse limit action, then the map

$$(\varphi_\lambda, u, \mathcal{H}) \rightarrow (\varphi_\lambda \circ \pi_\lambda^A, u, \mathcal{H})$$

is a bijection between $\mathcal{R}(G, \alpha^\lambda, A_\lambda)$ and $\mathcal{R}_\lambda(G, \alpha, A[\tau_\Gamma])$ (see, [J2]).

By [J2], if α is an inverse limit action, then $\mathcal{R}(G, \alpha, A[\tau_\Gamma])$ is non empty. From this result and Remark 3.6, we conclude that if α is strongly bounded, then $\mathcal{R}(G, \alpha, A[\tau_\Gamma])$ is non empty too. In the following proposition we give another proof for this result.

Proposition 4.4. *Let $(G, \alpha, A[\tau_\Gamma])$ be a pro- C^* -dynamical system such that α is strongly bounded. Then there is a covariant representation of $(G, \alpha, A[\tau_\Gamma])$.*

Proof. Let (φ, \mathcal{H}) be a representation of $A[\tau_\Gamma]$. Then there is $\lambda \in \Lambda$ such that $\|\varphi(a)\| \leq p_\lambda(a)$ for all $a \in A$. Let $a \in A$ and $\xi \in L^2(G, \mathcal{H})$. Since, there is $p_\mu \in \Gamma$ such that

$$\begin{aligned} \int_G \|\varphi(\alpha_{s^{-1}}(a))(\xi(s))\|^2 ds &\leq \int_G \|\varphi(\alpha_{s^{-1}}(a))\|^2 \|\xi(s)\|^2 ds \\ &\leq \int_G p_\lambda(\alpha_{s^{-1}}(a))^2 \|\xi(s)\|^2 ds \leq p_\mu(a)^2 \|\xi\|^2, \end{aligned}$$

the map $s \mapsto \varphi(\alpha_{s^{-1}}(a))(\xi(s))$ defines an element in $L^2(G, \mathcal{H})$. Therefore, there is $\tilde{\varphi}(a) \in L(L^2(G, \mathcal{H}))$ such that

$$\tilde{\varphi}(a)(\xi)(s) = \varphi(\alpha_{s^{-1}}(a))(\xi(s)).$$

In this way, we obtain a map $\tilde{\varphi} : A \rightarrow L(L^2(G, \mathcal{H}))$. Moreover, $\tilde{\varphi}$ is a continuous $*$ -morphism, and then $(\tilde{\varphi}, L^2(G, \mathcal{H}))$ is a representation of $A[\tau_\Gamma]$.

Let $(\lambda_G^\mathcal{H}, L^2(G, \mathcal{H}))$ be the unitary $*$ -representation of G on $L^2(G, \mathcal{H})$ given by $(\lambda_G^\mathcal{H})_t(\xi)(s) = \xi(t^{-1}s)$. It is easy to verify that $(\tilde{\varphi}, \lambda_G^\mathcal{H}, L^2(G, \mathcal{H}))$ is a covariant representation of $(G, \alpha, A[\tau_\Gamma])$. \square

Remark 4.5. *Let $(G, \alpha, A[\tau_\Gamma])$ be a pro- C^* -dynamical system. Suppose that α is strongly bounded. Then, for each representation (φ, \mathcal{H}) of $A[\tau_\Gamma]$, $\ker \tilde{\varphi} \subseteq \ker \varphi$. Indeed, if $\tilde{\varphi}(a) = 0$, then $\varphi(\alpha_s(a))(\xi(s)) = 0$ for all $s \in G$ and for all $\xi \in L^2(G, \mathcal{H})$, whence $\varphi(a)(\xi(e)) = 0$ for all $\xi \in L^2(G, \mathcal{H})$ and so $\varphi(a) = 0$.*

5. THE FULL PRO- C^* -CROSSED PRODUCT

Throughout this paper, $B[\tau_{\Gamma'}]$ is a pro- C^* -algebra with topology given by the family of C^* -seminorms $\Gamma' = \{q_\delta\}_{\delta \in \Delta}$. Let $(G, \alpha, A[\tau_\Gamma])$ be a pro- C^* -dynamical system.

Definition 5.1. *A covariant pro- C^* -morphism from $(G, \alpha, A[\tau_\Gamma])$ to a pro- C^* -algebra $B[\tau_{\Gamma'}]$ is a pair (φ, u) consisting of a pro- C^* -morphism $\varphi : A[\tau_\Gamma] \rightarrow M(B[\tau_{\Gamma'}])$ and a strict continuous group morphism $u : G \rightarrow \mathcal{U}(M(B[\tau_{\Gamma'}]))$ such that*

$$\varphi(\alpha_t(a)) = u_t \varphi(a) u_t^*$$

for all $t \in G$ and for all $a \in A$. A covariant pro- C^* -morphism (φ, u) from $(G, \alpha, A[\tau_\Gamma])$ to $B[\tau_{\Gamma'}]$ is nondegenerate if $[\varphi(A)B] = B[\tau_{\Gamma'}]$.

Theorem 5.2. *Let $(G, \alpha, A[\tau_\Gamma])$ be a pro- C^* -dynamical system. If α is strongly bounded, then there is a locally Hilbert space \mathcal{H} and a covariant pro- C^* -morphism (i_A, i_G) from $(G, \alpha, A[\tau_\Gamma])$ to $L(\mathcal{H})$. Moreover, i_A and i_G are injective.*

Proof. Let $\lambda \in \Lambda$. By Proposition 4.4, $\mathcal{R}_\lambda(G, \alpha, A[\tau_\Gamma])$ is non empty. Let $(\varphi^\lambda, u^\lambda, H_\lambda)$ be the direct sum of all equivalence classes of nondegenerate covariant representations $(\varphi, u, H_{\varphi, u})$ of $(G, \alpha, A[\tau_\Gamma])$, $(\varphi, u, H_{\varphi, u}) \in \mathcal{R}_\lambda(G, \alpha, A[\tau_\Gamma])$. Then $(\varphi^\lambda, u^\lambda, H_\lambda)$ is a nondegenerate covariant representation of $(G, \alpha, A[\tau_\Gamma])$ such that $\|\varphi^\lambda(a)\| \leq p_\lambda(a)$ for all $a \in A$.

Let $\mathcal{H}_\lambda = \oplus_{\mu \leq \lambda} H_\mu$. Then $\mathcal{H} = \lim_{\lambda \rightarrow} \mathcal{H}_\lambda$ is a locally Hilbert space. For $a \in A$, the map $i_A^\lambda(a) : \mathcal{H}_\lambda \rightarrow \mathcal{H}_\lambda$ defined by

$$i_A^\lambda(a) (\oplus_{\mu \leq \lambda} \xi_\mu) = \oplus_{\mu \leq \lambda} \varphi^\mu(a) \xi_\mu$$

is an element in $L(\mathcal{H}_\lambda)$ and $\|i_A^\lambda(a)\| \leq p_\lambda(a)$. Moreover, $i_A^\lambda(a^*) = i_A^\lambda(a)^*$ and $i_A^\lambda(ab) = i_A^\lambda(a) i_A^\lambda(b)$ for all $a, b \in A$. Clearly, $(i_A^\lambda(a))_\lambda$ is a direct system of bounded linear operators and $i_A(a) = \lim_{\lambda \rightarrow} i_A^\lambda(a)$ is an element $L(\mathcal{H})$ such that $i_A(a^*) = i_A(a)^*$ and $i_A(ab) = i_A(a) i_A(b)$ for all $a, b \in A$. Moreover,

$$p_{\lambda, L(\mathcal{H})}(i_A(a)) = \|i_A^\lambda(a)\| \leq p_\lambda(a)$$

for all $a \in A$ and for all $\lambda \in \Lambda$. Therefore, i_A is a pro- C^* -morphism.

For $t \in G$, the map $i_G^\lambda(t) : \mathcal{H}_\lambda \rightarrow \mathcal{H}_\lambda$ defined by

$$i_G^\lambda(t) (\oplus_{\mu \leq \lambda} \xi_\mu) = \oplus_{\mu \leq \lambda} u^\mu(t) \xi_\mu$$

is a unitary element in $L(\mathcal{H}_\lambda)$. Moreover, the map $t \mapsto i_G^\lambda(t)$ is a unitary $*$ -representation of G on \mathcal{H}_λ . Clearly, $(i_G^\lambda(t))_\lambda$ is a direct system of unitary operators, and then $i_G(t) = \lim_{\lambda \rightarrow} i_G^\lambda(t)$ is a unitary element $L(\mathcal{H})$. Moreover, $t \mapsto i_G(t)$ is a group morphism from G to the group of unitary operators on \mathcal{H} , and since for each $\xi \in \mathcal{H}$, the map $t \mapsto i_G(t) \xi$ from G to \mathcal{H} is continuous, $t \mapsto i_G(t)$ is a unitary $*$ -representation of G on \mathcal{H} . We have

$$\begin{aligned} i_A(\alpha_t(a)) (\oplus_{\mu \leq \lambda} \xi_\mu) &= i_A^\lambda(\alpha_t(a)) (\oplus_{\mu \leq \lambda} \xi_\mu) = \oplus_{\mu \leq \lambda} \varphi^\mu(\alpha_t(a)) (\xi_\mu) \\ &= \oplus_{\mu \leq \lambda} u^\mu(t) \varphi^\mu(a) u^\mu(t)^* (\xi_\mu) \\ &= i_G(t) i_A(a) i_G(t)^* (\oplus_{\mu \leq \lambda} \xi_\mu) \end{aligned}$$

for all $a \in A$, for all $t \in G$ and for all $\oplus_{\mu \leq \lambda} \xi_\mu \in \mathcal{H}_\lambda$, $\lambda \in \Lambda$, and so

$$i_A(\alpha_t(a)) = i_G(t) i_A(a) i_G(t)^*$$

for all $a \in A$ and for all $t \in G$.

Suppose that $i_A(a) = 0$. Then $i_A^\lambda(a) = 0$ for all $\lambda \in \Lambda$ and so $\varphi(a) = 0$ for all nondegenerate covariant representation $(\varphi, u, H_{\varphi, u})$ of $(G, \alpha, A[\tau_\Gamma])$. By Proposition 4.4 and Remark 4.5, $\psi(a) = 0$ for all representations ψ of A . Therefore, $p_\lambda(a) = 0$ for all $\lambda \in \Lambda$, and then $a = 0$.

Suppose that $i_G(t) = \text{id}_{\mathcal{H}}$. Then $i_G^\lambda(t) = \text{id}_{\mathcal{H}_\lambda}$ for all $\lambda \in \Lambda$, and so $u(t) = \text{id}_{H_{\varphi, u}}$ for all nondegenerate covariant representation $(\varphi, u, H_{\varphi, u})$ of $(G, \alpha, A[\tau_\Gamma])$, whence we deduce that $t = e$. \square

Proposition 5.3. *Let $(G, \alpha, A[\tau_\Gamma])$ be a pro- C^* -dynamical system. Then the following statements are equivalent.*

- (1) α is an inverse limit action.
- (2) There is a locally Hilbert space \mathcal{H} and a covariant pro- C^* -morphism (i_A, i_G) from $(G, \alpha, A[\tau_\Gamma])$ to $L(\mathcal{H})$ such that $p_{\lambda, L(\mathcal{H})}(i_A(a)) = p_\lambda(a)$ for all $\lambda \in \Lambda$ and $a \in A$.

Proof. (1) \Rightarrow (2) See [J3, Proposition 3.1] and [I, Theorem 5.1].

(2) \Rightarrow (1) From

$$i_A(\alpha_t(a)) = i_G(t) i_A(a) i_G(t)^*$$

for all $t \in G$ and for all $a \in A$, and taking into account that $i_G(t)$ is a unitary element in $L(\mathcal{H})$ for all $t \in G$, we deduce that

$$\begin{aligned} p_\lambda(\alpha_t(a)) &= p_{\lambda, L(\mathcal{H})}(i_A(\alpha_t(a))) = p_{\lambda, L(\mathcal{H})}(i_G(t) i_A(a) i_G(t)^*) \\ &= p_{\lambda, L(\mathcal{H})}(i_A(a)) = p_\lambda(a) \end{aligned}$$

for all $t \in G$, for all $a \in A$ and for all $t \in G$. Therefore, α is an inverse limit action. \square

If u is a strict continuous group morphism from G to $\mathcal{U}(M(B[\tau_{\Gamma'}]))$, then there is a $*$ -morphism $u : C_c(G) \rightarrow M(B[\tau_{\Gamma'}])$ given by $u(f) = \int_G f(s) u_s ds$, where ds denotes the Haar measure on G (see [J2]).

Definition 5.4. Let $(G, \alpha, A[\tau_\Gamma])$ be a pro- C^* -dynamical system. A pro- C^* -algebra, denoted by $G \times_\alpha A[\tau_\Gamma]$, together with a covariant pro- C^* -morphism (ι_A, ι_G) from $(G, \alpha, A[\tau_\Gamma])$ to $G \times_\alpha A[\tau_\Gamma]$ which verifies the following:

- (1) for each nondegenerate covariant representation $(\varphi, u, \mathcal{H})$ of $(G, \alpha, A[\tau_\Gamma])$, there is a nondegenerate representation (Φ, \mathcal{H}) of $G \times_\alpha A[\tau_\Gamma]$ such that $\overline{\Phi \circ \iota_A} = \varphi$ and $\overline{\Phi \circ \iota_G} = u$;
 - (2) $\overline{\text{span}\{\iota_A(a) \iota_G(f) ; a \in A, f \in C_c(G)\}} = G \times_\alpha A[\tau_\Gamma]$;
- is called the full pro- C^* -crossed product of $A[\tau_\Gamma]$ by α .

Remark 5.5. The covariant morphism (ι_A, ι_G) from the above definition is non-degenerate.

Proposition 5.6. Let $(G, \alpha, A[\tau_\Gamma])$ be a pro- C^* -dynamical system such that there is a full pro- C^* -crossed product of $A[\tau_\Gamma]$ by α and (φ, u) a nondegenerate covariant morphism from $(G, \alpha, A[\tau_\Gamma])$ to a pro- C^* -algebra $B[\tau_{\Gamma'}]$. Then there is a unique nondegenerate pro- C^* -morphism $\varphi \times u : G \times_\alpha A[\tau_\Gamma] \rightarrow M(B[\tau_{\Gamma'}])$ such that

$$\overline{\varphi \times u} \circ \iota_A = \varphi \text{ and } \overline{\varphi \times u} \circ \iota_G = u.$$

Moreover, the map $(\varphi, u) \rightarrow \varphi \times u$ is a bijection between nondegenerate covariant morphisms of $(G, \alpha, A[\tau_\Gamma])$ onto nondegenerate morphisms of $G \times_\alpha A[\tau_\Gamma]$.

Proof. Let $q_\delta \in \Gamma'$ and $(\psi_\delta, \mathcal{H})$ a faithful nondegenerate representation of B_δ . Then, $(\overline{\psi_\delta} \circ \overline{\pi_\delta^B} \circ \varphi, \overline{\psi_\delta} \circ \overline{\pi_\delta^B} \circ u, \mathcal{H})$ is a nondegenerate covariant representation of $(G, \alpha, A[\tau_\Gamma])$, and by Definition 5.4, there is a nondegenerate representation $(\phi_\delta, \mathcal{H})$ of $G \times_\alpha A[\tau_\Gamma]$ such that

$$\overline{\phi_\delta} \circ \iota_A = \overline{\psi_\delta} \circ \overline{\pi_\delta^B} \circ \varphi \text{ and } \overline{\phi_\delta} \circ \iota_G = \overline{\psi_\delta} \circ \overline{\pi_\delta^B} \circ u.$$

Let $\Phi_\delta = \overline{\psi_\delta^{-1}} \circ \phi_\delta$. Then Φ_δ is a nondegenerate pro- C^* -morphism from $G \times_\alpha A[\tau_\Gamma]$ to $M(B_\delta)$. Moreover, for $q_{\delta_1}, q_{\delta_2} \in \Gamma'$ with $q_{\delta_1} \geq q_{\delta_2}$, we have $\overline{\pi_{\delta_1 \delta_2}^B} \circ \Phi_{\delta_1} = \Phi_{\delta_2}$. Therefore, there is a nondegenerate pro- C^* -morphism $\varphi \times u : G \times_\alpha A[\tau_\Gamma] \rightarrow M(B[\tau_{\Gamma'}])$ such that

$$\overline{\pi_\delta^B} \circ \varphi \times u = \Phi_\delta$$

for all $q_\delta \in \Gamma'$. Moreover, $\overline{\varphi \times u} \circ \iota_A = \varphi$ and $\overline{\varphi \times u} \circ \iota_G = u$, and since $\{\iota_A(a) \iota_G(f) ; a \in A, f \in C_c(G)\}$ generates $G \times_\alpha A[\tau_\Gamma]$, $\varphi \times u$ is unique with the above properties.

Let $\Phi : G \times_\alpha A[\tau_\Gamma] \rightarrow M(B[\tau_{\Gamma'}])$ be a nondegenerate pro- C^* -morphism. Then $\varphi = \overline{\Phi} \circ \iota_A$ is a nondegenerate pro- C^* -morphism from $A[\tau_\Gamma]$ to $M(B[\tau_{\Gamma'}])$ and $u = \overline{\Phi} \circ \iota_G$ is a strict continuous morphism from G to $\mathcal{U}(M(B[\tau_{\Gamma'}]))$, since ι_G is a strict continuous morphism from G to $M(G \times_\alpha A[\tau_\Gamma])$ and $\overline{\Phi}$ is strongly continuous on the bounded subsets of $M(G \times_\alpha A[\tau_\Gamma])$. Moreover, (φ, u) is a nondegenerate covariant morphism from $A[\tau_\Gamma]$ to $B[\tau_{\Gamma'}]$, and $\varphi \times u = \Phi$. If (ψ, v) is another nondegenerate covariant morphism from $A[\tau_\Gamma]$ to $B[\tau_{\Gamma'}]$ such that $\psi \times v = \Phi$, then $\psi = \overline{\Phi} \circ \iota_A = \varphi$ and $v = \overline{\Phi} \circ \iota_G = u$. \square

The following corollary provides uniqueness of the full pro- C^* -crossed product by strong bounded actions.

Corollary 5.7. *Let $(G, \alpha, A[\tau_\Gamma])$ be a pro- C^* -dynamical system such that there is a full pro- C^* -crossed product of $A[\tau_\Gamma]$ by α . Then the full pro- C^* -crossed product of $A[\tau_\Gamma]$ by α is unique up to a pro- C^* -isomorphism.*

Proof. Let $B[\tau_{\Gamma'}]$ be a pro- C^* -algebra and (j_A, j_G) a covariant pro- C^* -morphism from $(G, \alpha, A[\tau_\Gamma])$ to $B[\tau_{\Gamma'}]$ which satisfy the relations (1) – (2) from Definition 5.4. Then, by Proposition 5.6, there is a nondegenerate pro- C^* -morphism $\Phi : G \times_\alpha A[\tau_\Gamma] \rightarrow M(B[\tau_{\Gamma'}])$ such that $\overline{\Phi} \circ \iota_A = j_A$ and $\overline{\Phi} \circ \iota_G = j_G$. Since $\{\iota_A(a) \iota_G(f), a \in A, f \in C_c(G)\}$ generates $G \times_\alpha A[\tau_\Gamma]$ and $\{j_A(a) j_G(f), a \in A, f \in C_c(G)\}$ generates $B[\tau_{\Gamma'}]$, $\Phi(G \times_\alpha A[\tau_\Gamma]) \subseteq B$.

In the same way, there is a pro- C^* -morphism $\Psi : B[\tau_{\Gamma'}] \rightarrow G \times_\alpha A[\tau_\Gamma]$ such that $\overline{\Psi} \circ j_A = \iota_A$ and $\overline{\Psi} \circ j_G = \iota_G$. From these facts and Definition 5.4(2), we deduce that $\Phi \circ \Psi = \text{id}_B$ and $\Psi \circ \Phi = \text{id}_{G \times_\alpha A[\tau_\Gamma]}$, and so Φ is a pro- C^* -isomorphism. \square

The following proposition relates the nondegenerate covariant representations of a pro- C^* -dynamical system $(G, \alpha, A[\tau_\Gamma])$ with the nondegenerate representations of the full pro- C^* -crossed product of $A[\tau_\Gamma]$ by α .

Proposition 5.8. *Let $(G, \alpha, A[\tau_\Gamma])$ be a pro- C^* -dynamical system such that there is the full pro- C^* -crossed product of $A[\tau_\Gamma]$ by α . There is a bijective correspondence between nondegenerate covariant representations of $(G, \alpha, A[\tau_\Gamma])$ and nondegenerate representations of $G \times_\alpha A[\tau_\Gamma]$.*

Proof. Let $(\varphi, u, \mathcal{H})$ be a nondegenerate covariant representation of $(G, \alpha, A[\tau_\Gamma])$. Then, by Definition 5.4, there is a representation $(\varphi \times u, \mathcal{H})$ such that $\overline{\varphi \times u} \circ \iota_A = \varphi$ and $\overline{\varphi \times u} \circ \iota_G = u$. Moreover, by Definition 5.4(2), $(\varphi \times u, \mathcal{H})$ is unique, and since φ is nondegenerate, it is nondegenerate too.

Let (Φ, \mathcal{H}) be a nondegenerate representation of $G \times_\alpha A[\tau_\Gamma]$. Then $(\overline{\Phi} \circ \iota_A, \overline{\Phi} \circ \iota_G, \mathcal{H})$ is a covariant representation of $(G, \alpha, A[\tau_\Gamma])$, and moreover, $(\overline{\Phi} \circ \iota_A) \times (\overline{\Phi} \circ \iota_G) = \Phi$. Since ι_A and Φ are nondegenerate, the net $\{\overline{\Phi}(\iota_A(e_i))\}_i$, where $\{e_i\}_i$ is an approximate unit of $A[\tau_\Gamma]$, converges strictly to $\text{id}_{\mathcal{H}}$, and so $\overline{\Phi} \circ \iota_A$ is nondegenerate.

Suppose that there is another nondegenerate covariant representation $(\varphi, u, \mathcal{H})$ of $(G, \alpha, A[\tau_\Gamma])$ such that $\varphi \times u = \Phi$. Then $\varphi = \overline{\varphi \times u} \circ \iota_A = \overline{\Phi} \circ \iota_A$ and $u = \overline{\varphi \times u} \circ \iota_G = \overline{\Phi} \circ \iota_G$. Therefore, the map $(\varphi, u, \mathcal{H}) \mapsto (\varphi \times u, \mathcal{H})$ is bijective. \square

Theorem 5.9. *Let $(G, \alpha, A[\tau_\Gamma])$ be a pro- C^* -dynamical system such that α is strongly bounded. Then, there is the full pro- C^* -crossed product of $A[\tau_\Gamma]$ by α .*

Proof. By Theorem 5.2, there is a locally Hilbert space \mathcal{H} and a covariant pro- C^* -morphism (i_A, i_G) from $A[\tau_\Gamma]$ to $L(\mathcal{H})$.

Let $B = \overline{\text{span}\{i_A(a) i_G(f); a \in A, f \in C_c(G)\}} \subseteq L(\mathcal{H})$. To show that B is a pro- C^* -algebra, we must show that B is closed under taking adjoints and multiplication. For this, since $B = \varprojlim_\lambda \pi_\lambda^{\mathcal{H}}(B)$ ([M, Chapter III, Theorem 3.1]), it is sufficient to show that for each $p_\lambda \in \Gamma$, $\pi_\lambda^{\mathcal{H}}(i_G(f) i_A(a))$ and $\pi_\lambda^{\mathcal{H}}(i_A(b) i_G(f) i_A(a) i_G(h))$ are elements in the closure of $\pi_\lambda^{\mathcal{H}}(B)$ in $L(\mathcal{H}_\lambda)$ for all $a, b \in A$ and for all $f, h \in C_c(G)$.

The map $s \rightarrow \pi_\lambda^A(f(s) \alpha_s(a))$ from G to A_λ defines an element in $C_c(G, A_\lambda)$, and so there is a net $\{\pi_\lambda^A(a_j) \otimes f_j\}_{j \in J}$ in $A_\lambda \otimes_{\text{alg}} C_c(G)$ with $\text{supp} f_j, \text{supp} f \subseteq K$ for some compact subset K , which converges uniformly to this map.

By [J4, Lemma 3.7],

$$\begin{aligned} \pi_\lambda^{\mathcal{H}}(i_G(f) i_A(a)) &= \int_G f(s) i^\lambda(s) ds \pi_\lambda^{\mathcal{H}}(i_A(a)) = \int_G f(s) \pi_\lambda^{\mathcal{H}}(i_G(s) i_A(a)) ds \\ &= \int_G f(s) \pi_\lambda^{\mathcal{H}}(i_A(\alpha_s(a)) i_G(s)) ds \\ &= \int_G f(s) i_A^\lambda(\alpha_s(a)) i_G^\lambda(s) ds \end{aligned}$$

and

$$\pi_\lambda^{\mathcal{H}}(i_A(a_j) i_G(f_j)) = \pi_\lambda^{\mathcal{H}}(i_A(a_j)) \int_G f_j(s) i_G^\lambda(s) ds = \int_G i_A^\lambda(a_j) f_j(s) i_G^\lambda(s) ds$$

for each $j \in J$. Then, we have

$$\begin{aligned} &\|\pi_\lambda^{\mathcal{H}}(i_G(f) i_A(a)) - \pi_\lambda^{\mathcal{H}}(i_A(a_j) i_G(f_j))\|_{L(\mathcal{H}_\lambda)} \\ &\leq \int_G \|f(s) i_A^\lambda(\alpha_s(a)) i_G^\lambda(s) - i_A^\lambda(a_j) f_j(s) i_G^\lambda(s)\|_{L(\mathcal{H}_\lambda)} ds \\ &\leq M \sup\{\|f(s) i_A^\lambda(\alpha_s(a)) i_G^\lambda(s) - i_A^\lambda(a_j) f_j(s) i_G^\lambda(s)\|_{L(\mathcal{H}_\lambda)}, s \in K\} \\ &= M \sup\{\|i_A^\lambda(f(s) \alpha_s(a) - f_j(s) a_j)\|_{L(\mathcal{H}_\lambda)} \|i_G^\lambda(s)\|_{L(\mathcal{H}_\lambda)}, s \in K\} \\ &\leq M \sup\{p_\lambda(f(s) \alpha_s(a) - f_j(s) a_j), s \in K\} \\ &= M \sup\{\|\pi_\lambda^A(f(s) \alpha_s(a)) - f_j(s) \pi_\lambda^A(a_j)\|_{A_\lambda}, s \in K\} \end{aligned}$$

for all $j \in J$, where $M = \int_K dg$, and so $\pi_\lambda^{\mathcal{H}}(i_G(f) i_A(a)) \in \overline{\pi_\lambda^{\mathcal{H}}(B)}$.

On the other hand,

$$\begin{aligned} &\|\pi_\lambda^{\mathcal{H}}(i_A(b) i_G(f) i_A(a) i_G(h)) - \pi_\lambda^{\mathcal{H}}(i_A(b a_j) i_G(f_j * h))\|_{L(\mathcal{H}_\lambda)} \\ &= \|\pi_\lambda^{\mathcal{H}}(i_A(b) i_G(f) i_A(a) i_G(h) - i_A(b) i_A(a_j) i_G(f_j) i_G(h))\|_{L(\mathcal{H}_\lambda)} \\ &\leq \|\pi_\lambda^{\mathcal{H}}(i_A(b)) \pi_\lambda^{\mathcal{H}}(i_G(f) i_A(a) - i_A(a_j) i_G(f_j)) \pi_\lambda^{\mathcal{H}}(i_G(h))\|_{L(\mathcal{H}_\lambda)} \\ &\leq \|\pi_\lambda^{\mathcal{H}}(i_A(b))\|_{L(\mathcal{H}_\lambda)} \|\pi_\lambda^{\mathcal{H}}(i_G(h))\|_{L(\mathcal{H}_\lambda)} \|\pi_\lambda^{\mathcal{H}}(i_G(f) i_A(a) - i_A(a_j) i_G(f_j))\|_{L(\mathcal{H}_\lambda)} \end{aligned}$$

whence, we deduce that $\pi_\lambda^{\mathcal{H}}(i_A(b) i_G(f) i_A(a) i_G(h)) \in \overline{\pi_\lambda^{\mathcal{H}}(B)}$. Thus, we showed that $\pi_\lambda^{\mathcal{H}}(i_G(f) i_A(a)), \pi_\lambda^{\mathcal{H}}(i_A(b) i_G(f) i_A(a) i_G(h)) \in \overline{\pi_\lambda^{\mathcal{H}}(B)}$ for each $\lambda \in \Lambda$, and so $i_G(f) i_A(a), i_A(b) i_G(f) i_A(a) i_G(h) \in B$. Therefore, B is a pro- C^* -algebra.

In the same manner, we show that for each $a \in A, i_A(a) i_A(b) i_G(f) \in B$ and $i_A(b) i_G(f) i_A(a) \in B$ for all $b \in A$ and for all $f \in C_c(G)$, and so $i_A(a) \in M(B)$.

From,

$$i_G(t)i_A(a)i_G(f) = \int_G f(s)i_A(\alpha_t(a))i_G(ts)ds \in B$$

and

$$i_G(f)i_A(a)i_G(t) = \int_G f(s)i_A(\alpha_s(a))i_G(st)ds \in B$$

for all $a \in A$, for all $f \in C_c(G)$ and for all $t \in G$, we deduce that $i_G(t) \in M(B)$ for all $t \in G$.

Let $(\psi, v, H_{\psi, v})$ be a nondegenerate covariant representation of $(G, \alpha, A[\tau_\Gamma])$. Then there is $(\varphi, u, H_{\varphi, u}) \in \mathcal{R}_\lambda(G, \alpha, A[\tau_\Gamma])$ such that $(\psi, v, H_{\psi, v})$ and $(\varphi, u, H_{\varphi, u})$ are unitarily equivalent. So there is a unitary operator $U : H_{\psi, v} \rightarrow H_{\varphi, u}$ such that $\psi(a) = U^*\varphi(a)U$ for all $a \in A$ and $v_t = U^*u_tU$ for all $t \in G$. The map $\Psi : L(\mathcal{H}) \rightarrow L(H_\lambda)$ given by

$$\Psi(T) = \pi_\lambda^{\mathcal{H}}(T)|_{H_\lambda}$$

is a representation of $L(\mathcal{H})$ on H_λ (see the proof of Theorem 5.2). From

$$\Psi(i_A(a))(H_{\varphi, u}) = i_A^\lambda(a)(H_{\varphi, u}) \subseteq H_{\varphi, u}$$

for all $a \in A$ and

$$\Psi(i_G(t))(H_{\varphi, u}) = i_G^\lambda(t)(H_{\varphi, u}) \subseteq H_{\varphi, u}$$

for all $t \in G$, and taking into account that B is generated by $\{i_A(a)i_G(f); a \in A, f \in C_c(G)\}$, we deduce that $\Psi(B)(H_{\varphi, u}) \subseteq H_{\varphi, u}$. Let $\Phi : B \rightarrow L(H_{\psi, v})$ given by

$$\Phi(b) = U^*\Psi(b)|_{H_{\varphi, u}}U.$$

Clearly, Φ is a nondegenerate representation of B on $H_{\psi, v}$,

$$\overline{\Phi}(i_A(a)) = U^*\Psi(i_A(a))|_{H_{\varphi, u}}U = U^*i_A^\lambda(a)|_{H_{\varphi, u}}U = U^*\varphi(a)U = \psi(a)$$

for all $a \in A$, and

$$\overline{\Phi}(i_G(t)) = U^*\Psi(i_G(t))|_{H_{\varphi, u}}U = U^*i_G^\lambda(t)|_{H_{\varphi, u}}U = U^*u_tU = v_t$$

for all $t \in G$. \square

Remark 5.10. *The index of the family of seminorms which gives the topology on the full pro- C^* -crossed product of $A[\tau_\Gamma]$ by α is the same with the index of the family of seminorms which gives the topology on $A[\tau_\Gamma]$.*

Proposition 5.11. *Let $(G, \alpha, A[\tau_\Gamma])$ be a pro- C^* -dynamical system such that α is an inverse limit action. Then for each $\lambda \in \Lambda$, the C^* -algebra $(G \times_\alpha A[\tau_\Gamma])_\lambda$ is isomorphic to the full C^* -crossed product of A_λ by α^λ .*

Proof. By Theorem 5.2, Proposition 5.3 and Corollary 5.7, there is a C^* -morphism $i_{A_\lambda} : A_\lambda \rightarrow M((G \times_\alpha A[\tau_\Gamma])_\lambda)$ such that $i_{A_\lambda} \circ \pi_\lambda^A = \overline{\pi_\lambda^{G \times_\alpha A[\tau_\Gamma]}} \circ i_A$. Using the fact that α is an inverse limit action, it is easy to check that $(i_{A_\lambda}, \overline{\pi_\lambda^{G \times_\alpha A[\tau_\Gamma]}} \circ i_G)$ is a covariant C^* -morphism from $(G, \alpha^\lambda, A_\lambda)$ to $(G \times_\alpha A[\tau_\Gamma])_\lambda$. Moreover,

$$\begin{aligned} & \overline{\text{span}\{i_{A_\lambda}(\pi_\lambda^A(a))\overline{\pi_\lambda^{G \times_\alpha A[\tau_\Gamma]}}(i_G(f)); a \in A, f \in C_c(G)\}} \\ &= \overline{\text{span}\{\overline{\pi_\lambda^{G \times_\alpha A[\tau_\Gamma]}}(i_A(a)i_G(f)); a \in A, f \in C_c(G)\}} \\ &= \overline{\pi_\lambda^{G \times_\alpha A[\tau_\Gamma]}}(G \times_\alpha A[\tau_\Gamma]) = (G \times_\alpha A[\tau_\Gamma])_\lambda. \end{aligned}$$

Let $(\varphi, u, \mathcal{H})$ be a nondegenerate covariant representation of $(G, \alpha^\lambda, A_\lambda)$. Then $(\varphi \circ \pi_\lambda^A, u, \mathcal{H})$ is a nondegenerate covariant representation of $(G, \alpha, A[\tau_\Gamma])$ and by Definition 5.4, there is a nondegenerate representation (Φ, \mathcal{H}) of $G \times_\alpha A[\tau_\Gamma]$ such that $\overline{\Phi} \circ i_A = \varphi \circ \pi_\lambda^A$ and $\overline{\Phi} \circ i_G = u$. Moreover, by the proof of Theorem 5.9,

$$\|\Phi(b)\| \leq p_{\lambda, G \times_\alpha A[\tau_\Gamma]}(b)$$

for all $b \in G \times_\alpha A[\tau_\Gamma]$. Therefore, there is the C^* -morphism $\Phi_\lambda : (G \times_\alpha A[\tau_\Gamma])_\lambda \rightarrow L(\mathcal{H})$ such that $\Phi_\lambda \circ \pi_\lambda^{G \times_\alpha A[\tau_\Gamma]} = \Phi$. Moreover, $(\Phi_\lambda, \mathcal{H})$ is a nondegenerate representation of $(G \times_\alpha A[\tau_\Gamma])_\lambda$ such that

$$\overline{\Phi}_\lambda \circ i_{A_\lambda} = \varphi \text{ and } \overline{\Phi}_\lambda \circ \left(\pi_\lambda^{G \times_\alpha A[\tau_\Gamma]} \circ i_G \right) = u.$$

Thus, we showed that $(G \times_\alpha A[\tau_\Gamma])_\lambda$ is isomorphic to $G \times_{\alpha^\lambda} A_\lambda$. \square

Corollary 5.12. *Let $(G, \alpha, A[\tau_\Gamma])$ be a pro- C^* -dynamical system such that α is an inverse limit action. Then the pro- C^* -algebras $G \times_\alpha A[\tau_\Gamma]$ and $\lim_{\leftarrow \lambda} G \times_{\alpha^\lambda} A_\lambda$ are isomorphic.*

Remark 5.13. *If $(G, \alpha, A[\tau_\Gamma])$ is a pro- C^* -dynamical system such that α is an inverse limit action, then the notion of full pro- C^* -crossed product in the sense of Definition 5.4 coincides to the notion of full crossed product introduced by [P2, J4].*

Let $A[\tau_\Gamma]$ and $B[\tau_{\Gamma'}]$ be two pro- C^* -algebras. For each $p_\lambda \in \Gamma$ and $q_\delta \in \Gamma'$, the map $t_{p_\lambda, q_\delta} : A[\tau_\Gamma] \otimes_{\text{alg}} B[\tau_{\Gamma'}] \rightarrow [0, \infty)$ given by

$$t_{p_\lambda, q_\delta}(z) = \sup\{\|\varphi \circ \pi_{p_\lambda, q_\delta}(z)\|; \varphi \text{ is a } * \text{-representation of } A_\lambda \otimes_{\text{alg}} B_j\},$$

where $\pi_{p_\lambda, q_\delta}(a \otimes b) = \pi_\lambda^A(a) \otimes \pi_\delta^B(b)$, defines a C^* -seminorm on the algebraic tensor product $A[\tau_\Gamma] \otimes_{\text{alg}} B[\tau_{\Gamma'}]$. The completion of $A[\tau_\Gamma] \otimes_{\text{alg}} B[\tau_{\Gamma'}]$ with respect to the topology given by the family of C^* -seminorms $\{t_{p_\lambda, q_\delta}; p_\lambda \in \Gamma, q_\delta \in \Gamma'\}$ is a pro- C^* -algebra, denoted by $A[\tau_\Gamma] \otimes_{\max} B[\tau_{\Gamma'}]$, and called the maximal or projective tensor product of the pro- C^* -algebras $A[\tau_\Gamma]$ and $B[\tau_{\Gamma'}]$ (see [F, Chapter VII]). Moreover, for each $p_\lambda \in \Gamma$ and $q_\delta \in \Gamma'$, the C^* -algebras $(A[\tau_\Gamma] \otimes_{\max} B[\tau_{\Gamma'}])_{(\lambda, \delta)}$ and $A_\lambda \otimes_{\max} B_\delta$ are isomorphic.

Remark 5.14. *The trivial action of G on $A[\tau_\Gamma]$ is an inverse limit action, and so the full pro- C^* -crossed product of $A[\tau_\Gamma]$ by the trivial action is isomorphic to $A[\tau_\Gamma] \otimes_{\max} C^*(G)$ [J2, Corollary 1.3.9].*

Let $(G, \alpha, A[\tau_\Gamma])$ be a pro- C^* -dynamical system such that α is strongly bounded and let $B[\tau_{\Gamma'}]$ be a pro- C^* -algebra. Then $t \mapsto (\alpha \otimes \text{id})_t$, where $(\alpha \otimes \text{id})_t(a \otimes b) = \alpha_t(a) \otimes b$, is a strong bounded action of G on $A[\tau_\Gamma] \otimes_{\max} B[\tau_{\Gamma'}]$.

The following theorem gives an "associativity" between \times_α and \otimes_{\max} .

Theorem 5.15. *Let $(G, \alpha, A[\tau_\Gamma])$ be a pro- C^* -dynamical system such that α is strongly bounded and let $B[\tau_{\Gamma'}]$ be a pro- C^* -algebra. Then the pro- C^* -algebras $G \times_{\alpha \otimes \text{id}} (A[\tau_\Gamma] \otimes_{\max} B[\tau_{\Gamma'}])$ and $(G \times_\alpha A[\tau_\Gamma]) \otimes_{\max} B[\tau_{\Gamma'}]$ are isomorphic.*

Proof. Let $\rho_{G \times_\alpha A[\tau_\Gamma]} : G \times_\alpha A[\tau_\Gamma] \rightarrow M((G \times_\alpha A[\tau_\Gamma]) \otimes_{\max} B[\tau_{\Gamma'}])$ and $\rho_B : B[\tau_{\Gamma'}] \rightarrow M((G \times_\alpha A[\tau_\Gamma]) \otimes_{\max} B[\tau_{\Gamma'}])$ be the canonical maps. Then $\overline{\rho_{G \times_\alpha A[\tau_\Gamma]}} \circ \iota_A : A[\tau_\Gamma] \rightarrow M((G \times_\alpha A[\tau_\Gamma]) \otimes_{\max} B[\tau_{\Gamma'}])$ and $\rho_B : B[\tau_{\Gamma'}] \rightarrow M((G \times_\alpha A[\tau_\Gamma]) \otimes_{\max} B[\tau_{\Gamma'}])$ are nondegenerate pro- C^* -morphisms with commuting ranges.

Let $j_{G \times_{\alpha \otimes \text{id}} A[\tau_\Gamma] \otimes_{\max} B[\tau_{\Gamma'}]} = \overline{\rho_{G \times_\alpha A[\tau_\Gamma]}} \circ \iota_A \otimes \rho_B$ and $j_G = \overline{\rho_{G \times_\alpha A[\tau_\Gamma]}} \circ \iota_G$. A simple calculus shows that $(j_{G \times_{\alpha \otimes \text{id}} A[\tau_\Gamma] \otimes_{\max} B[\tau_{\Gamma'}]}, j_G)$ is a nondegenerate covariant

pro- C^* -morphism from $(G, \alpha \otimes \text{id}, A[\tau_\Gamma] \otimes_{\max} B[\tau_{\Gamma'}])$ to $M((G \times_\alpha A[\tau_\Gamma]) \otimes_{\max} B[\tau_{\Gamma'}])$. Moreover, from

$$\begin{aligned} & j_{G \times_\alpha \otimes \text{id} A[\tau_\Gamma] \otimes_{\max} B[\tau_{\Gamma'}]}(a \otimes b) j_G(f) \\ &= \overline{\rho_{G \times_\alpha A[\tau_\Gamma]}(\iota_A(a)) \rho_B(b)} \overline{\rho_{G \times_\alpha A[\tau_\Gamma]}(\iota_G(f))} \\ &= \overline{\rho_{G \times_\alpha A[\tau_\Gamma]}(\iota_A(a))} \overline{\rho_{G \times_\alpha A[\tau_\Gamma]}(\iota_G(f))} \rho_B(b) \\ &= \rho_{G \times_\alpha A[\tau_\Gamma]}(\iota_A(a) \iota_G(f)) \rho_B(b) \end{aligned}$$

for all $a \in A$, for all $b \in B$ and for all $f \in C_c(G)$, and taking into account that $\{\iota_A(a) \iota_G(f); a \in A, f \in C_c(G)\}$ generates $G \times_\alpha A[\tau_\Gamma]$ and $\{\rho_{G \times_\alpha A[\tau_\Gamma]}(z) \rho_B(b); z \in G \times_\alpha A[\tau_\Gamma], b \in B\}$ generates $(G \times_\alpha A[\tau_\Gamma]) \otimes_{\max} B[\tau_{\Gamma'}]$, we conclude that

$$\begin{aligned} & \overline{\text{span}\{j_{G \times_\alpha \otimes \text{id} A[\tau_\Gamma] \otimes_{\max} B[\tau_{\Gamma'}]}(a \otimes b) j_G(f); a \in A, b \in B, f \in C_c(G)\}} \\ &= (G \times_\alpha A[\tau_\Gamma]) \otimes_{\max} B[\tau_{\Gamma'}]. \end{aligned}$$

Let $(\varphi, u, \mathcal{H})$ be a nondegenerate covariant representation of $(G, \alpha \otimes \text{id}, A[\tau_\Gamma] \otimes_{\max} B[\tau_{\Gamma'}])$. Then (φ, \mathcal{H}) is a nondegenerate representation of $A[\tau_\Gamma] \otimes_{\max} B[\tau_{\Gamma'}]$, and so there is a nondegenerate representation $(\varphi_{(\lambda, \delta)}, \mathcal{H})$ of $A_\lambda \otimes_{\max} B_\delta$ such that $\varphi_{(\lambda, \delta)} \circ \pi_{(\lambda, \delta)}^{A[\tau_\Gamma] \otimes_{\max} B[\tau_{\Gamma'}]} = \varphi$. Let $(\varphi_\lambda, \mathcal{H})$ and $(\varphi_\delta, \mathcal{H})$ be the nondegenerate representations of A_λ , respectively B_δ with commuting ranges such that $\varphi_{(\lambda, \delta)}(a \otimes b) = \varphi_\lambda(a) \varphi_\delta(b)$ for all $a \in A_\lambda$ and $b \in B_\delta$. Then $(\varphi_\lambda \circ \pi_\lambda^A, u, \mathcal{H})$ is a nondegenerate covariant representation of $(G, \alpha, A[\tau_\Gamma])$, and so there is a nondegenerate representation (Φ_1, \mathcal{H}) of $G \times_\alpha A[\tau_\Gamma]$ such that $\overline{\Phi_1} \circ \iota_A = \varphi_\lambda \circ \pi_\lambda^A$ and $\overline{\Phi_1} \circ \iota_G = u$. It is easy to check that (Φ_1, \mathcal{H}) and (Φ_2, \mathcal{H}) , where $\Phi_2 = \varphi_\delta \circ \pi_\delta^B$, are nondegenerate representations of $G \times_\alpha A[\tau_\Gamma]$ respectively $B[\tau_{\Gamma'}]$ with commuting ranges. Let (Φ, \mathcal{H}) be the nondegenerate representation of $(G \times_\alpha A[\tau_\Gamma]) \otimes_{\max} B[\tau_{\Gamma'}]$ given by $\Phi(z \otimes b) = \Phi_1(z) \Phi_2(b)$. Then

$$\begin{aligned} & \overline{\Phi}(j_{G \times_\alpha \otimes \text{id} A[\tau_\Gamma] \otimes_{\max} B[\tau_{\Gamma'}]}(a \otimes b)) \\ &= \overline{\Phi}\left(\overline{\rho_{G \times_\alpha A[\tau_\Gamma]}(\iota_A(a)) \rho_B(b)}\right) = \overline{\Phi_1}((\iota_A(a)) \overline{\Phi_2}(\rho_B(b))) \\ &= (\varphi_\lambda \circ \pi_\lambda^A(a)) (\varphi_\delta \circ \pi_\delta^B(b)) = \varphi_{(\lambda, \delta)}(\pi_\lambda^A(a) \otimes \pi_\delta^B(b)) \\ &= \varphi_{(\lambda, \delta)} \circ \pi_{(\lambda, \delta)}^{A[\tau_\Gamma] \otimes_{\max} B[\tau_{\Gamma'}]}(a \otimes b) = \varphi(a \otimes b) \end{aligned}$$

for all $a \in A$ and $b \in B$, and

$$\overline{\Phi}(j_G(f)) = \overline{\Phi}\left(\overline{\rho_{G \times_\alpha A[\tau_\Gamma]} \circ \iota_G(f)}\right) = \overline{\Phi_1}(\iota_G(f)) = u(f)$$

for all $f \in C_c(G)$. Therefore, by Definition 5.4 and Corollary 5.7, the pro- C^* -algebras $G \times_\alpha \otimes \text{id} A[\tau_\Gamma] \otimes_{\max} B[\tau_{\Gamma'}]$ and $(G \times_\alpha A[\tau_\Gamma]) \otimes_{\max} B[\tau_{\Gamma'}]$ are isomorphic. \square

Definition 5.16. We say that $(G, \alpha, A[\tau_\Gamma])$ and $(G, \beta, B[\tau_{\Gamma'}])$ are conjugate if there is a pro- C^* -isomorphism $\varphi : A[\tau_\Gamma] \rightarrow B[\tau_{\Gamma'}]$ such that $\varphi \circ \alpha_t = \beta_t \circ \varphi$ for all $t \in G$.

Remark 5.17. If $(G, \alpha, A[\tau_\Gamma])$ and $(G, \beta, B[\tau_{\Gamma'}])$ are conjugate and α is strongly bounded, then β is strongly bounded too.

Proposition 5.18. Let $(G, \alpha, A[\tau_\Gamma])$ and $(G, \beta, B[\tau_{\Gamma'}])$ be two pro- C^* -dynamical systems such that α and β are strongly bounded. If $(G, \alpha, A[\tau_\Gamma])$ and $(G, \beta, B[\tau_{\Gamma'}])$

are conjugate, then the full pro- C^* -crossed products associated to these pro- C^* -dynamical systems are isomorphic.

Proof. Let $\varphi : A[\tau_\Gamma] \rightarrow B[\tau_{\Gamma'}]$ be a pro- C^* -isomorphism such that $\varphi \circ \alpha_t = \beta_t \circ \varphi$ for all $t \in G$. It is easy to check that $(\iota_B \circ \varphi, \iota_{G,B})$ is a nondegenerate covariant morphism from $(G, \alpha, A[\tau_\Gamma])$ to $G \times_\beta B[\tau_{\Gamma'}]$, where $(\iota_B, \iota_{G,B})$ is the covariant morphism from $(G, \beta, B[\tau_{\Gamma'}])$ to $G \times_\beta B[\tau_{\Gamma'}]$ which defines the full pro- C^* -crossed product of $B[\tau_{\Gamma'}]$ by β . Then, by Proposition 5.6, there is a nondegenerate pro- C^* -morphism $\Phi : G \times_\alpha A[\tau_\Gamma] \rightarrow M(G \times_\beta B[\tau_{\Gamma'}])$ such that $\overline{\Phi} \circ \iota_A = \iota_B \circ \varphi$ and $\overline{\Phi} \circ \iota_{G,A} = \iota_{G,B}$. Moreover, using Definition 5.4, it is easy to check that $\Phi(G \times_\alpha A[\tau_\Gamma]) \subseteq G \times_\beta B[\tau_{\Gamma'}]$. In the same manner, we obtain a nondegenerate pro- C^* -morphism $\Psi : G \times_\beta B[\tau_{\Gamma'}] \rightarrow M(G \times_\alpha A[\tau_\Gamma])$ such that $\overline{\Psi} \circ \iota_B = \iota_A \circ \varphi^{-1}$ and $\overline{\Psi} \circ \iota_{G,B} = \iota_{G,A}$.

From

$$(\Phi \circ \Psi)(\iota_B(b) \iota_{G,B}(f)) = \Phi(\iota_A \circ \varphi^{-1}(b) \iota_{G,A}(f)) = \iota_B(b) \iota_{G,B}(f)$$

and

$$(\Psi \circ \Phi)(\iota_A(a) \iota_{G,A}(f)) = \Psi(\iota_B \circ \varphi(a) \iota_{G,B}(f)) = \iota_A(a) \iota_{G,A}(f)$$

for all $b \in B[\tau_{\Gamma'}]$, $a \in A[\tau_\Gamma]$ and $f \in C_c(G)$ and Definition 5.4, we deduce that Φ and Ψ are pro- C^* -isomorphisms. \square

Corollary 5.19. *Let $(G, \alpha, A[\tau_\Gamma])$ be a pro- C^* -dynamical system such that α is strongly bounded.*

- (1) *Pro- C^* -algebras $G \times_\alpha A[\tau_\Gamma]$ and $G \times_\alpha A[\tau_{\Gamma^G}]$ are isomorphic.*
- (2) *$A[\tau_\Gamma]$ is isomorphic to a pro- C^* -subalgebra of $M(G \times_\alpha A[\tau_\Gamma])$.*

6. THE REDUCED PRO- C^* -CROSSED PRODUCT

Let $A[\tau_\Gamma]$ and $B[\tau_{\Gamma'}]$ be two pro- C^* -algebras. For each $p_\lambda \in \Gamma$ and $q_\delta \in \Gamma'$, the map $\vartheta_{p_\lambda, q_\delta} : A[\tau_\Gamma] \otimes_{\text{alg}} B[\tau_{\Gamma'}] \rightarrow [0, \infty)$ given by

$$\vartheta_{p_\lambda, q_\delta}(z) = \sup\{\|(\varphi \otimes \psi)(z)\|; \varphi \in \mathcal{R}_\lambda(A[\tau_\Gamma]), \psi \in \mathcal{R}_\delta(B[\tau_{\Gamma'}])\}$$

defines a C^* -seminorm on the algebraic tensor product $A[\tau_\Gamma] \otimes_{\text{alg}} B[\tau_{\Gamma'}]$. The completion of $A[\tau_\Gamma] \otimes_{\text{alg}} B[\tau_{\Gamma'}]$ with respect to the topology given by the family of C^* -seminorms $\{\vartheta_{p_\lambda, q_\delta}; p_\lambda \in \Gamma, q_\delta \in \Gamma'\}$ is a pro- C^* -algebra, denoted by $A[\tau_\Gamma] \otimes B[\tau_{\Gamma'}]$, and called the minimal or injective tensor product of the pro- C^* -algebras $A[\tau_\Gamma]$ and $B[\tau_{\Gamma'}]$ (see [F, Chapter VII]). Moreover, for each $p_\lambda \in \Gamma$ and $q_\delta \in \Gamma'$, the C^* -algebras $(A[\tau_\Gamma] \otimes_{\min} B[\tau_{\Gamma'}])_{(\lambda, \delta)}$ and $A_\lambda \otimes_{\min} B_\delta$ are isomorphic.

Let $(G, \alpha, A[\tau_\Gamma])$ be a pro- C^* -dynamical system such that α is strongly bounded. Since α is strongly bounded, for each $a \in A$, the map $t \mapsto \alpha_{t^{-1}}(a)$ defines an element in $C_b(G, A[\tau_\Gamma])$, the pro- C^* -algebra of all bounded continuous functions from G to $A[\tau_\Gamma]$, and so there is a map $\tilde{\alpha} : A[\tau_\Gamma] \rightarrow C_b(G, A[\tau_\Gamma])$ given by $\tilde{\alpha}(a)(t) = \alpha_{t^{-1}}(a)$.

Lemma 6.1. *Let $(G, \alpha, A[\tau_\Gamma])$ be a pro- C^* -dynamical system such that α is strongly bounded. Then $\tilde{\alpha}$ is a nondegenerate faithful pro- C^* -morphism from $A[\tau_\Gamma]$ to $M(A[\tau_\Gamma] \otimes_{\min} C_0(G))$ with closed range. Moreover, if α is an inverse limit action, then $\tilde{\alpha}$ is an inverse limit pro- C^* -morphism.*

Proof. Clearly, $\tilde{\alpha}$ is a $*$ -morphism. For each $p_\lambda \in \Gamma$, there is $p_\mu \in \Gamma$ such that

$$p_\lambda(a) = p_\lambda(\alpha_e(a)) \leq \sup\{p_\lambda(\alpha_t(a)); t \in G\} = p_{\lambda, C_b(G, A[\tau_\Gamma])}(\tilde{\alpha}(a)) \leq p_\mu(a)$$

for all $a \in A$. Therefore, $\tilde{\alpha}$ is an injective pro- C^* -morphism with closed range. By [J2, p. 76], $C_b(G, A[\tau_\Gamma])$ can be identified to a pro- C^* -subalgebra of $M(A[\tau_\Gamma] \otimes C_0(G))$, and then $\tilde{\alpha}$ can be regarded as a pro- C^* -morphism from $A[\tau_\Gamma]$ to $M(A[\tau_\Gamma] \otimes C_0(G))$.

To show that $\tilde{\alpha}$ is nondegenerate, let $\{e_i\}_{i \in I}$ be an approximate unit for $A[\tau_\Gamma]$. In the same manner as in [V, Proposition 5.1.5], we show that $\{\tilde{\alpha}(e_i)\}_{i \in I}$ is strictly convergent. Indeed, let $a \in A$, $f \in C_c(G)$ and $p_\lambda \in \Gamma$. Then

$$\begin{aligned} & p_{\lambda, C_b(G, A[\tau_\Gamma])} (\tilde{\alpha}(e_i)(a \otimes f) - a \otimes f) \\ &= \sup \{ p_\lambda(\alpha_{t^{-1}}(e_i)af(t) - af(t)); t \in G \} \\ &\leq \|f\|_\infty \sup \{ p_\lambda(\alpha_{t^{-1}}(e_i\alpha_t(a) - \alpha_t(a))); t \in \text{supp}(f) \} \\ &\leq \|f\|_\infty \sup \{ p_\mu(e_i\alpha_t(a) - \alpha_t(a)); t \in \text{supp}(f) \} \end{aligned}$$

for some $p_\mu \in \Gamma$. For each $i \in I$, consider the function $f_i : G \rightarrow \mathbb{C}$, $f_i(t) = p_\mu(e_i\alpha_t(a) - \alpha_t(a))$. Clearly, $\{f_i\}_{i \in I}$ is a net of continuous functions on G which is uniformly bounded and equicontinuous. Then, by Arzelà–Ascoli’s theorem, it is uniformly convergent on compact subsets of G . Therefore, $\{\tilde{\alpha}(e_i)\}_{i \in I}$ is strictly convergent, and so the pro- C^* -morphism $\tilde{\alpha}$ is nondegenerate.

Suppose that $\alpha_t = \lim_{\leftarrow \lambda} \alpha_t^\lambda$ for each $t \in G$. Then $(\tilde{\alpha}^\lambda)_\lambda$ is an inverse system of C^* -morphisms and $\tilde{\alpha} = \lim_{\leftarrow \lambda} \tilde{\alpha}^\lambda$. \square

Let $\varphi : A[\tau_\Gamma] \rightarrow M(B[\tau_{\Gamma'}])$ be a nondegenerate pro- C^* -morphism and let $M : C_0(G) \rightarrow L(L^2(G))$ be the representation by multiplication operators. Then there is a nondegenerate pro- C^* -morphism $\varphi \otimes M : A[\tau_\Gamma] \otimes_{\min} C_0(G) \rightarrow M(B[\tau_{\Gamma'}] \otimes_{\min} \mathcal{K}(L^2(G)))$ such that $(\varphi \otimes M)(a \otimes f) = \varphi(a) \otimes M_f$, and since $\tilde{\alpha}$ is a nondegenerate pro- C^* -morphism from $A[\tau_\Gamma]$ to $M(A[\tau_\Gamma] \otimes_{\min} C_0(G))$, $\tilde{\varphi} = \overline{\varphi \otimes M} \circ \tilde{\alpha}$ is a nondegenerate pro- C^* -morphism from $A[\tau_\Gamma]$ to $M(B[\tau_{\Gamma'}] \otimes_{\min} \mathcal{K}(L^2(G)))$.

Let $\lambda_G : G \rightarrow \mathcal{U}(L^2(G))$ be the left representation of G on $L^2(G)$ given by $(\lambda_G)_t(\xi)(s) = \xi(t^{-1}s)$. Then $1 \otimes \lambda_G : G \rightarrow \mathcal{U}(M(B[\tau_{\Gamma'}] \otimes_{\min} \mathcal{K}(L^2(G))))$, where $(1 \otimes \lambda_G)_t(b \otimes \xi)(s) = b\xi(t^{-1}s)$, is a strict continuous group morphism from G to $\mathcal{U}(M(B[\tau_{\Gamma'}] \otimes_{\min} \mathcal{K}(L^2(G))))$, and $(\tilde{\varphi}, 1 \otimes \lambda_G)$ is a nondegenerate covariant morphism of $(G, \alpha, A[\tau_\Gamma])$ to $B[\tau_{\Gamma'}] \otimes_{\min} \mathcal{K}(L^2(G))$. By Proposition 5.6, there is a unique nondegenerate pro- C^* -morphism $\tilde{\varphi} \times (1 \otimes \lambda_G) : G \times_\alpha A[\tau_\Gamma] \rightarrow M(B[\tau_{\Gamma'}] \otimes_{\min} \mathcal{K}(L^2(G)))$ such that $\tilde{\varphi} \times (1 \otimes \lambda_G) \circ \iota_A = \tilde{\varphi}$ and $\tilde{\varphi} \times (1 \otimes \lambda_G) \circ \iota_G = 1 \otimes \lambda_G$.

If $\varphi = \text{id}_A$, the nondegenerate pro- C^* -morphism $\text{id}_A \times (1 \otimes \lambda_G) : G \times_\alpha A[\tau_\Gamma] \rightarrow M(A[\tau_\Gamma] \otimes_{\min} \mathcal{K}(L^2(G)))$ is denoted by Λ_A^G . It is easy to check that $\tilde{\varphi} \times (1 \otimes \lambda_G) = \overline{\varphi \otimes \text{id}_{\mathcal{K}(L^2(G))}} \circ \Lambda_A^G$.

If α is an inverse limit action, $\alpha_t = \lim_{\leftarrow \lambda} \alpha_t^\lambda$ for each $t \in G$, then it is easy to check that Λ_A^G is an inverse limit pro- C^* -morphism, $\Lambda_A^G = \lim_{\leftarrow \lambda} \Lambda_{A_\lambda}^G$.

Definition 6.2. *The reduced pro- C^* -crossed product of $A[\tau_\Gamma]$ by α is the pro- C^* -subalgebra $G \times_{\alpha, r} A[\tau_\Gamma]$ of $M(A[\tau_\Gamma] \otimes_{\min} \mathcal{K}(L^2(G)))$ generated by the range of Λ_A^G .*

Remark 6.3. *From*

$$\Lambda_A^G(\iota_A(a) \iota_G(f)) = (\overline{\text{id}_A \otimes M} \circ \tilde{\alpha})(a)(1 \otimes \lambda_G)(f) = \tilde{\alpha}(a)(1 \otimes \lambda_G(f))$$

for all $a \in A$ and for all $f \in C_c(G)$, and taking into account that $G \times_\alpha A[\tau_\Gamma]$ is generated by $\{\iota_A(a) \iota_G(f); a \in A, f \in C_c(G)\}$, we conclude that $G \times_{\alpha, r} A[\tau_\Gamma]$ is the

pro- C^* -subalgebra of $M(A[\tau_\Gamma] \otimes_{\min} \mathcal{K}(L^2(G)))$ generated by $\{\tilde{\alpha}(a)(1 \otimes \lambda_G(f)); a \in A, f \in C_c(G)\}$.

Remark 6.4. If α is an inverse limit action, $\alpha_t = \lim_{\leftarrow \lambda} \alpha_t^\lambda$ for each $t \in G$, then

$$G \times_{\alpha, r} A[\tau_\Gamma] = \overline{\Lambda_A^G(G \times_\alpha A[\tau_\Gamma])} = \lim_{\leftarrow \lambda} \overline{\Lambda_{A_\lambda}^G(G \times_{\alpha^\lambda} A_\lambda)} = \lim_{\leftarrow \lambda} G \times_{\alpha^\lambda, r} A_\lambda$$

and moreover, for each $p_\lambda \in \Gamma$, the C^* -algebras $(G \times_{\alpha, r} A[\tau_\Gamma])_\lambda$ and $G \times_{\alpha^\lambda, r} A_\lambda$ are isomorphic.

Remark 6.5. Since the trivial action of a locally compact group G on a pro- C^* -algebra $A[\tau_\Gamma]$ is an inverse limit action, the reduced pro- C^* -crossed product of $A[\tau_\Gamma]$ by the trivial action is the inverse limit of the reduced crossed products of A_λ by the trivial action, and so it is isomorphic to the pro- C^* -algebra $A[\tau_\Gamma] \otimes_{\min} C_r^*(G)$.

Let $(G, \alpha, A[\tau_\Gamma])$ be a pro- C^* -dynamical system such that α is strongly bounded and let $B[\tau_{\Gamma'}]$ be a pro- C^* -algebra. Then $t \mapsto (\alpha \otimes \text{id})_t$, where $(\alpha \otimes \text{id})_t(a \otimes b) = \alpha_t(a) \otimes b$, is a strong bounded action of G on $A[\tau_\Gamma] \otimes B[\tau_{\Gamma'}]$.

The following theorem gives an "associativity" between $\times_{\alpha, r}$ and \otimes_{\min} .

Theorem 6.6. Let $(G, \alpha, A[\tau_\Gamma])$ be a pro- C^* -dynamical system such that α is strongly bounded and let $B[\tau_{\Gamma'}]$ be a pro- C^* -algebra. Then the pro- C^* -algebras $G \times_{\alpha \otimes \text{id}, r} (A[\tau_\Gamma] \otimes_{\min} B[\tau_{\Gamma'}])$ and $(G \times_{\alpha, r} A[\tau_\Gamma]) \otimes_{\min} B[\tau_{\Gamma'}]$ are isomorphic.

Proof. The map $\text{id}_A \otimes \sigma_{B, \mathcal{K}(L^2(G))} : A[\tau_\Gamma] \otimes_{\min} B[\tau_{\Gamma'}] \otimes_{\min} \mathcal{K}(L^2(G)) \rightarrow A[\tau_\Gamma] \otimes_{\min} \mathcal{K}(L^2(G)) \otimes_{\min} B[\tau_{\Gamma'}]$ given by

$$\text{id}_A \otimes \sigma_{B, \mathcal{K}(L^2(G))}(a \otimes b \otimes T) = a \otimes T \otimes b$$

is a pro- C^* -isomorphism. Moreover, $\text{id}_A \otimes \sigma_{B, \mathcal{K}(L^2(G))}$ is an inverse limit of C^* -isomorphisms. From

$$\overline{\text{id}_A \otimes \sigma_{B, \mathcal{K}(L^2(G))}} \left(\widetilde{\alpha \otimes \text{id}}(a \otimes b)(1 \otimes \lambda_G(f)) \right) = \tilde{\alpha}(a)(1_{M(A[\tau_\Gamma])} \otimes \lambda_G(f) \otimes b)$$

for all $a \in A$, for all $b \in B$ and for all $f \in C_c(G)$, and Remark 6.3, we deduce that

$$\overline{\text{id}_A \otimes \sigma_{B, \mathcal{K}(L^2(G))}}|_{G \times_{\alpha \otimes \text{id}, r} A[\tau_\Gamma] \otimes_{\min} B[\tau_{\Gamma'}]}$$

is a pro- C^* -isomorphism from $G \times_{\alpha \otimes \text{id}, r} (A[\tau_\Gamma] \otimes_{\min} B[\tau_{\Gamma'}])$ onto $(G \times_{\alpha, r} A[\tau_\Gamma]) \otimes_{\min} B[\tau_{\Gamma'}]$. Therefore, the pro- C^* -algebras $(G \times_{\alpha, r} A[\tau_\Gamma]) \otimes_{\min} B[\tau_{\Gamma'}]$ and $G \times_{\alpha \otimes \text{id}, r} (A[\tau_\Gamma] \otimes_{\min} B[\tau_{\Gamma'}])$ are isomorphic. \square

Proposition 6.7. Let $(G, \alpha, A[\tau_\Gamma])$ and $(G, \beta, B[\tau_{\Gamma'}])$ be two pro- C^* -dynamical systems such that α and β are strongly bounded. If $(G, \alpha, A[\tau_\Gamma])$ and $(G, \beta, B[\tau_{\Gamma'}])$ are conjugate, then the reduced pro- C^* -crossed products associated to these pro- C^* -dynamical systems are isomorphic.

Proof. Let $\varphi : A[\tau_\Gamma] \rightarrow B[\tau_{\Gamma'}]$ be a pro- C^* -isomorphism such that $\varphi \circ \alpha_t = \beta_t \circ \varphi$ for all $t \in G$. It is easy to check that $\overline{\varphi \otimes \text{id}_{\mathcal{K}(L^2(G))}} \circ \tilde{\alpha} = \tilde{\beta} \circ \varphi$. From

$$\overline{\varphi \otimes \text{id}_{\mathcal{K}(L^2(G))}}(\tilde{\alpha}(a)(1 \otimes \lambda_G(f))) = \tilde{\beta}(\varphi(a))(1 \otimes \lambda_G(f))$$

for all $a \in A$ and for all $f \in C_c(G)$, and taking into account that

$$\overline{\text{span}\{\tilde{\alpha}(a)(1 \otimes \lambda_G(f)); a \in A, f \in C_c(G)\}} = G \times_{\alpha, r} A[\tau_\Gamma]$$

and

$$\overline{\text{span}\{\tilde{\beta}(a)(1 \otimes \lambda_G(f)); a \in B, f \in C_c(G)\}} = G \times_{\beta,r} B[\tau_{\Gamma'}],$$

we conclude that $\Phi_1 = \overline{\varphi \otimes \text{id}_{\mathcal{K}(L^2(G))}}|_{G \times_{\alpha,r} A[\tau_{\Gamma}]}$ is a pro- C^* -morphism from $G \times_{\alpha,r} A[\tau_{\Gamma}]$ to $G \times_{\beta,r} B[\tau_{\Gamma'}]$.

In the same manner, we conclude that $\Phi_2 = \overline{\varphi^{-1} \otimes \text{id}_{\mathcal{K}(L^2(G))}}|_{G \times_{\beta,r} B[\tau_{\Gamma}]}$ is a pro- C^* -morphism from $G \times_{\beta,r} B[\tau_{\Gamma}]$ to $G \times_{\alpha,r} A[\tau_{\Gamma'}]$. Moreover, $\Phi_1 \circ \Phi_2 = \text{id}_{G \times_{\beta,r} B[\tau_{\Gamma}]}$ and $\Phi_2 \circ \Phi_1 = \text{id}_{G \times_{\alpha,r} A[\tau_{\Gamma}]}$, since

$$\Phi_1 \circ \Phi_2 \left(\tilde{\beta}(b)(1 \otimes \lambda_G(f)) \right) = \tilde{\beta}(b)(1 \otimes \lambda_G(f))$$

for all $b \in B$ and for all $f \in C_c(G)$ and

$$\Phi_2 \circ \Phi_1 (\tilde{\alpha}(a)(1 \otimes \lambda_G(f))) = \tilde{\alpha}(a)(1 \otimes \lambda_G(f))$$

for all $a \in A$ and for all $f \in C_c(G)$. Therefore, the pro- C^* -algebras $G \times_{\alpha,r} A[\tau_{\Gamma}]$ and $G \times_{\beta,r} B[\tau_{\Gamma'}]$ are isomorphic. \square

Corollary 6.8. *Let $(G, \alpha, A[\tau_{\Gamma}])$ be a pro- C^* -dynamical systems such that α is strongly bounded. Then the pro- C^* -algebras $G \times_{\alpha,r} A[\tau_{\Gamma}]$ and $G \times_{\alpha,r} A[\tau_{\Gamma G}]$ are isomorphic.*

Remark 6.9. *If α is an action of an amenable locally compact group G on a C^* -algebra A , then the C^* -morphism Λ_A^G is injective and the full crossed product of A by α is isomorphic to the reduced crossed product of A by α . If α is an inverse limit action of an amenable locally compact group G on a pro- C^* -algebra $A[\tau_{\Gamma}]$, $\alpha_t = \lim_{\leftarrow \lambda} \alpha_t^\lambda$ for each $t \in G$, then $\Lambda_A^G = \lim_{\leftarrow \lambda} \Lambda_{A_\lambda}^G$, and so Λ_A^G is an injective pro- C^* -morphism with closed range. Therefore, if G is amenable and α is an inverse limit action, then the full pro- C^* -crossed product of $A[\tau_{\Gamma}]$ by α is isomorphic to the reduced pro- C^* -crossed product of $A[\tau_{\Gamma}]$ by α .*

Proposition 6.10. *Let $(G, \alpha, A[\tau_{\Gamma}])$ be a pro- C^* -dynamical system such that α is strongly bounded. If G is amenable then the full pro- C^* -crossed product of $A[\tau_{\Gamma}]$ by α is isomorphic to the reduced pro- C^* -crossed product of $A[\tau_{\Gamma}]$ by α .*

Proof. It follows from Corollaries 5.20 and 6.8, and Remark 6.9. \square

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